MAT224 - LEC5101 - Lecture 6
Review and Test Prep

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Overview: Review and Test Prep

Test Content Announcement on Quercus!

Plan for today:

- We're going to slow down, take stock of what we've already learned, and practice solving problems.
- Explicit review of general problem solving strategies, which hopefully you already remember being mentioned in passing
- After each one, I'll give an example problem, you'll try to solve it following the general strategy, and I'll take up the solution.

Before we begin: some general test preperation advice:

- Study. Pratice solving problems. Organize your understanding.
- Sleep; Study in advance so you have lots of time.
- Eat (maybe not immidiately before the test).
- Relax your mind for 10-20 minutes just before writing the test.
- Look over your notes one last time just before that.

Writing tests, solving problems:

First, a few more general tips about writing tests:

- Read the questions carefully.
- Don't panic if you get stuck on a problem; move on.
- Try to at least look at all the questions before the test ends.
- Answer the true-false questions!

General problem solving plan:

(1) Draw the picture.Or several of different possible examples.

(2) Write the definitions.(Important to know them perfectly)

(3) Write what you need to show, and 'plug in' the definitions!

(4) Show it!(This is usually the hard part, but not always)

Tip: Put the question in context:What topic(s) are you being tested on? What concepts do you know that seem relevant? Or simply: what theorems did you learn about the definitions?

Discussion: Showing sets are equal

General Problem: Let U, W be some sets, and prove $U = W$:
General Answer: Apply the steps:

(1) Picture: Can you picture U, W? Maybe in an example?
(2) Definitions: e.g. $U = \{\mathbf{v} \in V | \ \mathbf{v} \text{ has property } P_U\}$ and
$W = \{\mathbf{v} \in V | \ \mathbf{v} \text{ has property } P_W\}$.
(3) WTS: Let $\mathbf{v} \in V$. Then we want to show $\mathbf{v}$ has property P_U if and only if $\mathbf{v}$ has property P_W.
(4) Is P_U the same as P_W? If non-trivial, show P_U implies P_W and then vice versa (i.e. show $U \subset W$ and $W \subset U$).

Problem: Let $V = P_2(\mathbb{R})$, $S = \{1 + 2x - x^2, 1 + x + x^2\} \subset V$, $W = \{a + bx + cx^2 | -3a + 2b + c = 0\}$. Show $\text{Span}(S) = W$.

(1) Draw a picture, or picture this in your head. See why it's true?
(2) Write down definition of Span(S). (W is clear here.)
(3) WTS: what? Is this obvious? What about one direction of it? What is the other direction? What intuition/tools to use?

Discussion: Showing elements linearly independent

General Problem: Show $S = \{\mathbf{x}_1, ..., \mathbf{x}_n\} \subset V$ is lin. indep.

(1) Try to picture these elements in V.Are any of them colinear?
(2) Write the definition of linear independence.
(3) Suppose $a_1\mathbf{x}_1 + ... + a_n\mathbf{x}_n = 0$, then show $a_1 = 0$, $a_2 = 0$, ...
(4) Often this is immidiate. Otherwise, use the hypotheses!

Problem: Show that $\{1 + x, 1 - x, 1 + x^2\}$ is linearly independent.

(1) Draw a picture, or in your head! See why it's true?
(2) Write down the definition linear independence for three vectors
(3) Plug in the actual values. Write resulting equations
(4) Solve these equations to show $a_1 = 0$, $a_2 = 0$ and $a_3 = 0$.

Problem: Suppose $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ are lin. indep. and $n \geq 3$. Show that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are lin. indep.

(1) Picture. Why's it true? (2) Definition. Write it!
(3) WTS: write it! (4) Prove it. Use the hypothesis!

Discussion: Linear Independence continued

General solution, repeated:

(1) Try to picture these elements in V. Are any of them colinear?
(2) Write the definition of linear independence.
(3) Suppose $a_1\mathbf{x}_1 + ... + a_n\mathbf{x}_n = 0$, then show $a_1 = 0$, $a_2 = 0$, ...
(4) Often this is immidiate. Otherwise, use the hypotheses!

Problem: Let $U, W \subset V$ subspaces with bases $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ and $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$. If $U \cap W = \{\mathbf{0}\}$, $\{\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{w}_1, ..., \mathbf{w}_m\}$ is linearly independent.

(1) Picture. Why's it true? (2) Definition. Write it!
(3) WTS: write it! (4) Prove it. Use the hypothesis!

Problem: Suppose $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in V$ and that $\{\mathbf{x}_1, \mathbf{x}_2\}$, $\{\mathbf{x}_2, \mathbf{x}_3\}$ and $\{\mathbf{x}_1, \mathbf{x}_3\}$ are linearly independent. Is $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ lin. indep.?

(1) Picture examples. Especially when correct answer not given!
(2) Write the definition. This will be needed either way!
(3) Write what you need to show about potential counterexample.
(4) Show this holds in your counterexample.

Discussion: Intersection, Sum, Direct Sum

Definition: Let $U, W \subset V$ be subspaces. Recall the subspaces

$U \cap W = \{\mathbf{v} \in V | \mathbf{v} \in U \text{ and } \mathbf{v} \in W\}$

$U + W = \{\mathbf{v} = \mathbf{u} + \mathbf{w} | \mathbf{u} \in U \text{ and } \mathbf{w} \in W\}$

New: $U \oplus W$ is the subspace $U + W$ when $U \cap W = \{\mathbf{0}\}$.

Proposition: $\dim U + W = \dim U + \dim W - \dim(U \cap W)$

Example: Let $V = \mathbb{R}^3$, $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2 = x_3 = 0\}$ and $W = \text{Span}(\{(0, 1, 1), (0, 1, -1)\})$. Then $V = U \oplus W$.

Problem: Does $V = U \oplus W$ in each of the following?

(1) $V = \mathbb{R}^3$, $U = \{(x_1, x_2, x_3) | x_1 = x_2\}$ and $W = \text{Span}(0, 1, 0)$

(2) $V = \mathbb{R}^4$, $U = \{(x_1, x_2, x_3, x_4) | x_1 = x_4, x_2 = x_3\}$, and $W = \text{Span}\{(0, 1, 0, 1), (1, 2, 1, 2)\}$

Problem Let $V = U \oplus W$, and $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$, $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$ bases for U and W. Show that $\{\mathbf{u}_1, ..., \mathbf{u}_n, \mathbf{w}_1, ..., \mathbf{w}_m\}$ is a basis for V.

Discussion: Overview of 'Word Problems'

Major Key: We know that after choosing a basis, the vector spaces $M_{m\times n}(\mathbb{R})$ and $P_n(\mathbb{R})$ the just $\mathbb{R}^{nm}$ and $\mathbb{R}^{n+1}$. Problems like this are just our version of 'word problems' in highschool math.

Most questions in this course is about existence or uniqueness of solutions to linear equations!

If a problem has weird conditions, translate into equations!

Problem: Let $W = \{A \in M_{n\times n}(\mathbb{R}) | \text{tr}(A) = 0\}$. What's dim W?

(Recall: $\text{tr}(A) = a_{11} + a_{22} + ... + a_{nn}$)

Solution:

- This is just one linear equation on an n^2 dimensional vector space $V = M_{n\times n}(\mathbb{R})$.
- By rank-nullity (or equivalent formulation):

  dim solutions = dim V - number of independent equations

  $\dim\ker T = \dim V - \dim \operatorname{im} T$
- Thus, $\dim W = n^2 - 1$.

Discussion: more word problems

Let $V = P_n(\mathbb{R})$. A polynomial $p(x) = a_0 + a_1 x + ... + a_n x^n$ is called palindromic if $a_k = a_{n-k}$ for each k.

Problem: Calculate the dimension of the subspace W of palindromic polynomials in $P_3(\mathbb{R})$.

- Draw a 'picture' Think of examples
- Translate into abstract equations
- Solve the problem

Similarly, $p(x)$ is called pseudo-palindromic if there exists $s \in \mathbb{R}$ such that $a_k - a_{n-k} = s$ for each k.

Problem: Calculate the dimension of the subspace W of pseudo-palindromic polynomials in $P_3(\mathbb{R})$.

- Think of some examples Draw a 'picture'
- Translate into abstract equations
- Solve the problem